Refinements of the trace inequality of Belmega, Lasaulce and Debbah

Shigeru Furuichi[1]*and Minghua Lin[2]†
[1]Department of Computer Science and System Analysis,
College of Humanities and Sciences, Nihon University,
3-25-40, Sakurajyousui, Setagaya-ku, Tokyo, 156-8550, Japan
[2]Department of Mathematics and Statistics,
University of Regina, Regina, Saskatchewan, Canada S4S 0A2

Abstract. In this short paper, we show a certain matrix trace inequality and then give a refinement of the trace inequality proven by Belmega, Lasaulce and Debbah. In addition, we give an another improvement of their trace inequality.

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1 Introduction

Recently, E.-V.Belmega, S.Lasaulce and M.Debbah obtained the following elegant trace inequality for positive definite matrices.

Theorem 1.1 **([1])** *For positive definite matrices A, B and positive semidefinite matrices C, D, we have*

$$Tr[(A-B)(B^{-1}-A^{-1})+(C-D)\left\{(B+D)^{-1}-(A+C)^{-1}\right\}]\geq 0. \tag{1}$$

In this short paper, we first prove a certain trace inequality for products of matrices, and then as its application, we give a simple proof of (1). At the same time, our alternative proof gives a refinement and of Theorem 1.1. An another improvement of the Theorem 1.1 is also considered at the end of the paper.

2 Main results

In this section, we prove the following theorem.

Theorem 2.1 *For positive definite matrices A, B and positive semidefinite matrices C, D, we have*

$$\begin{aligned}&Tr[(A-B)(B^{-1}-A^{-1})+(C-D)\left\{(B+D)^{-1}-(A+C)^{-1}\right\}]\\&\geq |Tr[(C-D)(B+D)^{-1}(A-B)(A+C)^{-1}]|.\end{aligned} \tag{2}$$

To prove this theorem, we need a few lemmas.

*E-mail:furuichi@chs.nihon-u.ac.jp
†E-mail:lin243@uregina.ca

Lemma 2.2 **([1])** *For positive definite matrices A, B and positive semidefinite matrices C, D, and Hermitian matrix X, we have*

$$Tr[XA^{-1}XB^{-1}] \geq Tr[X(A+C)^{-1}X(B+D)^{-1}].$$

Lemma 2.3 *For any matrices X and Y, we have*

$$Tr[X^*X] + Tr[Y^*Y] \geq 2|Tr[X^*Y]|.$$

Proof: Since $Tr[X^*X] \geq 0$, by the fact that the arithmetical mean is greater than the geometrical mean and Cauchy-Schwarz inequality, we have

$$\frac{Tr[X^*X] + Tr[Y^*Y]}{2} \geq \sqrt{Tr[X^*X]Tr[Y^*Y]} \geq |Tr[X^*Y]|.$$

■

Theorem 2.4 *For Hermitian matrices X_1, X_2 and positive semidefinite matrices S_1, S_2, we have*

$$Tr[X_1S_1X_1S_2] + Tr[X_2S_1X_2S_2] \geq 2|Tr[X_1S_1X_2S_2]|.$$

Proof: Applying Lemma 2.3, we have

$$\begin{aligned}
&Tr[X_1S_1X_1S_2] + Tr[X_2S_1X_2S_2] \\
&\quad = Tr[(S_2^{1/2}X_1S_1^{1/2})(S_1^{1/2}X_1S_2^{1/2})] + Tr[(S_2^{1/2}X_2S_1^{1/2})(S_1^{1/2}X_2S_2^{1/2})] \\
&\quad \geq 2|Tr[(S_2^{1/2}X_1S_1^{1/2})(S_1^{1/2}X_2S_2^{1/2})]| \\
&\quad = 2|Tr[X_1S_1X_2S_2]|.
\end{aligned}$$

■

Remark 2.5 *Theorem 2.4 can be regarded as a kind of the generalization of Proposition 1.1 in [2].*

Proof of Theorem 2.1: By Lemma 2.2, we have

$$\begin{aligned}
Tr[(A-B)(B^{-1}-A^{-1})] &= Tr[(A-B)B^{-1}(A-B)A^{-1}] \\
&\geq Tr[(A-B)(A+C)^{-1}(A-B)(B+D)^{-1}] \\
&= Tr[(A-B)(B+D)^{-1}(A-B)(A+C)^{-1}].
\end{aligned}$$

Thus the left hand side of the inequality (2) can be bounded from below:

$$\begin{aligned}
&Tr[(A-B)(B^{-1}-A^{-1}) + (C-D)\left\{(B+D)^{-1}-(A+C)^{-1}\right\}] \\
&\geq Tr[(A-B)(B+D)^{-1}(A-B)(A+C)^{-1} + (C-D)(B+D)^{-1}(C-D)(A+C)^{-1}] \\
&\quad +Tr[(C-D)(B+D)^{-1}(A-B)(A+C)^{-1}] \\
&\geq 2|Tr[(C-D)(B+D)^{-1}(A-B)(A+C)^{-1}]| \\
&\quad +Tr[(C-D)(B+D)^{-1}(A-B)(A+C)^{-1}] \qquad (3)
\end{aligned}$$

Throughout the process of the above, Theorem 2.4 was used in the second inequality. Since we have the following equation,

$$\begin{aligned}
&Tr[(C-D)(B+D)^{-1}(A-B)(A+C)^{-1}] \\
&\quad = Tr[(C-D)(B+D)^{-1}] - Tr[(C-D)(A+C)^{-1}] \\
&\quad -Tr[(C-D)(B+D)^{-1}(C-D)(A+C)^{-1}]
\end{aligned}$$

we have $Tr[(C-D)(B+D)^{-1}(A-B)(A+C)^{-1}] \in \mathbb{R}$. Therefore we have

$$(3) \geq |Tr[(C-D)(B+D)^{-1}(A-B)(A+C)^{-1}]|.$$

■

3 An another improvement of the inequality (1)

In this section, we show the following trace inequality.

Theorem 3.1 *For positive definite matrices A, B and positive semidefinite matrices C, D, we have*

$$Tr[(A-B)(B^{-1}-A^{-1})+4(C-D)\left\{(B+D)^{-1}-(A+C)^{-1}\right\}]\geq 0. \tag{4}$$

To prove this theorem, we use the following lemmas, which are proven by the similar way of Lemma 2.3 and Theorem 2.4 in the previous section.

Lemma 3.2 *For any matrices X and Y, any positive real numbers a and b, we have*

$$a\cdot Tr[X^*X]+b\cdot Tr[Y^*Y]\geq 2\sqrt{ab}\cdot |Tr[X^*Y]|.$$

Applying this lemma, we have the following lemma.

Lemma 3.3 *For Hermitian matrices X_1, X_2, positive semidefinite matrices S_1, S_2 and any positive real numbers a and b, we have*

$$a\cdot Tr[X_1S_1X_1S_2]+b\cdot Tr[X_2S_1X_2S_2]\geq 2\sqrt{ab}\cdot |Tr[X_1S_1X_2S_2]|.$$

Proof of Theorem 3.1: By the similar way to the proof of Theorem 2.1, applying Lemma 3.2 as $a=1$ and $b=4$, the left hand side of the inequality of (4) can be bounded from the below:

$$\begin{aligned}
&Tr[(A-B)(B^{-1}-A^{-1})+4(C-D)\left\{(B+D)^{-1}-(A+C)^{-1}\right\}]\\
&\geq Tr[(A-B)(B+D)^{-1}(A-B)(A+C)^{-1}+4(C-D)(B+D)^{-1}(C-D)(A+C)^{-1}]\\
&\quad +Tr[4(C-D)(B+D)^{-1}(A-B)(A+C)^{-1}]\\
&\geq 4|Tr[(C-D)(B+D)^{-1}(A-B)(A+C)^{-1}]|\\
&\quad +4\cdot Tr[(C-D)(B+D)^{-1}(A-B)(A+C)^{-1}]\geq 0,
\end{aligned}$$

since $Tr[(C-D)(B+D)^{-1}(A-B)(A+C)^{-1}]\in\mathbb{R}$. ■

Remark 3.4 *Here we note that we have $Tr[(A-B)(B^{-1}-A^{-1})]\geq 0$. However we have the possibility that $Tr[(C-D)\left\{(B+D)^{-1}-(A+C)^{-1}\right\}]$ takes a negative value. Therefore Theorem 3.1 is an improvement of Theorem 1.1.*

Corollary 3.5 *For positive definite matrices A, B, positive semidefinite matrices C, D and positive real number r, we have*

$$Tr[(A-B)(B^{-1}-A^{-1})+4(C-D)\left\{(rB+D)^{-1}-(rA+C)^{-1}\right\}]\geq 0. \tag{5}$$

Proof: Put $A=rA_1$ and $B=rB_1$ for positive definite matrices A_1 and B_1, in Theorem 3.1. ■

Remark 3.6 *In the case of $r=2$ in Corollary 3.5, the inequality (5) corresponds to the scalar inequality:*

$$(\alpha-\beta)\left(\frac{1}{4\beta}-\frac{1}{4\alpha}\right)+(\gamma-\delta)\left(\frac{1}{2\beta+\delta}-\frac{1}{2\alpha+\gamma}\right)\geq 0$$

for positive real numbers α and β, nonnegative real numbers γ and δ.

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